SOME PROPOSITIONS CONCERNING THE GEOMETRIC REPRESENTATION OF IMAGINARIES.

By DR. MAXIME BÔCHER, Cambridge, Mass.

The present paper contains a few propositions concerning the geometric representation of the roots of equations. The principal proposition contained in Sec. 1 is due to Lucas who gave a mechanical proof for it. I do not know whether the geometrical proof here given has been published before. I have found no reference to the propositions of Sec. 2.

1. Let $f(z) = 0$ represent an algebraic equation of the nth degree in z, and $f'(z) = 0$ its first derivative with regard to z. We shall make no assumption concerning the reality of the coefficients of $f(z) = 0$, but shall not consider the case in which all the roots of this equation are real. We will denote these n roots by $z_1, z_2, \ldots, z_n$ and the $n - 1$ roots of the derived equation by $z'_1, z'_2, \ldots, z'_{n-1}$. All of these quantities will in general be complex, and we will represent them in the ordinary way by points in a plane. The object of the present paper is to discuss the geometric relation which the system of $n - 1$ points $z'_1, \ldots, z'_{n-1}$ bears to the system of n points $z_1, \ldots, z_n$.

We will first prove the following generalization of Rolle's theorem:

*If a convex polygon be formed whose vertices are the outermost of the points $z_1, \ldots, z_n$, this polygon will include all the points of the system $z'_1, \ldots, z'_{n-1}$.**

To prove this we will prolong any one of the sides AB of the polygon indefinitely in both directions, and take that direction upon it as positive in which it must be described in order that the rest of the polygon should lie to its left. We will prove that no point z which lies to the right of this line can be a root of the equation $f'(z) = 0$. Let us denote by θ the angle measured from the positive direction of the axis of reals to the positive direction of AB, and by φ the argument of the complex quantity $f(\bar{z})$. Writing, now,

$$f'(\bar{z}) = \frac{f(\bar{z})}{\bar{z} - z_1} + \frac{f(\bar{z})}{\bar{z} - z_2} + \ldots + \frac{f(\bar{z})}{\bar{z} - z_n},$$

it is easy to see that the argument of each of the terms in the second member lies between $\varphi - \theta$ and $\varphi - \theta - 180°$. These terms will therefore be represented by points lying to the right of the line through the origin, which makes an angle of $\varphi - \theta$ with the axis of reals. Their sum can therefore not be zero,

* Exceptions will occur not merely when all the points $z_1, \ldots, z_n$ lie on a straight line, this being practically the case of real roots; but also when $f(z) = 0$ has multiple roots, for then some of the points $z'_1, \ldots, z'_{n-1}$ will lie not *within* the polygon but at its vertices.

and z cannot be a root of $f'(z) = 0$. All the points $z'_1, \ldots, z'_{n-1}$ must then lie on the same side of the line AB as the polygon itself. The line AB was however any side of the polygon, whence it follows that the points $z'_1, \ldots, z'_{n-1}$ all lie within the polygon.

The theorem just proved is however merely a qualitative one, and we naturally wish to determine quantitatively the relation between the two systems of points.

Writing the equation $f(z) = 0$ in the form

$$z^n + nA_1 z^{n-1} + \frac{n(n-1)}{2!} A_2 z^{n-2} + \ldots + n A_{n-1} z + A_n = 0,$$

we have the relation

$$\frac{z_1 + z_2 + \ldots + z_n}{n} = - A_1 = \frac{z'_1 + z'_2 + \ldots + z'_{n-1}}{n-1},$$

from which we can infer at once the simple proposition,

The centre of gravity of the points $z_1, \ldots, z_n$, coincides with the centre of gravity of the points $z'_1, \ldots, z'_{n-1}$.

In Sec. 2 I have attempted to supplement this obvious proposition by others of a similar nature.

2. We will first consider the special case in which $f(z) = 0$ is the general cubic equation

$$z^3 + 3A_1 z^2 + 3A_2 z + A_3 = 0;$$

and ask ourselves, How are the two points z'_1, z'_2 related to the triangle whose vertices are the points z_1, z_2, z_3? A natural supposition is that the two points z'_1, z'_2 are the foci of some conic simply related to this triangle. The centre of such a conic must, according to the closing proposition of Sec. 1, lie at the centre of gravity of the triangle. Now one conic whose centre lies at the centre of gravity of a triangle is the ellipse tangent to the sides of the triangle at their middle points, or, as it is often called from one of its properties, the maximum ellipse inscribed in the triangle. We are thus naturally led up to the following proposition :—

The points z'_1, z'_2 are the foci of the maximum ellipse inscribed in the triangle whose vertices are z_1, z_2, z_3.

In order to prove this theorem I will first show that the points z'_1, z'_2 possess a certain property of the foci in question. This property depends upon the fact (see Salmon's Conic Sections, p. 182) that a tangent to an ellipse from a point P makes the same angle with the line joining P to one of the foci as the line joining P to the other focus makes with the other tangent from P. Here we will take as the point P any of the vertices of the triangle, for instance

z_3. F_1 and F_2 being the foci in question, our proposition tells us that the angles $z_1 z_3 F_1$ and $F_2 z_3 z_2$ are equal. Now, we can prove directly that the angles $z_1 z_3 z'_1$ and $z'_2 z_3 z_2$ are equal, for they are the arguments of the complex quantities $\frac{z'_1 - z_3}{z_1 - z_3}$ and $\frac{z_2 - z_3}{z'_2 - z_3}$. In order to prove them equal we have, then, merely to show that the quotient of these two quantities is real and positive, and this quotient is

$$\frac{(z'_1 - z_3)(z'_2 - z_3)}{(z_1 - z_3)(z_2 - z_3)} = \frac{z'_1 z'_2 - z_3(z'_1 + z'_2) + z_3^2}{z_1 z_2 - z_3(z_1 + z_2) + z_3^2} = \frac{A_2 + 2A_1 z_3 + z_3^2}{3A_2 + 6A_1 z_3 + 3z_3^2} = \frac{1}{3}.$$

Now with the points z'_1, z'_2 as foci describe an ellipse tangent to one side of the triangle. Then, by the closing proposition of Sec. 1 its centre will coincide with the centre of gravity of the triangle, and by the proposition just proved the ellipse will also be tangent to the other two sides of the triangle. Therefore, since only one conic can be drawn with a given point as centre and tangent to three given lines, this ellipse must be the maximum ellipse and the points z'_1, z'_2 must be the foci of the maximum ellipse.*

A part of the proposition thus proved may without difficulty be extended as follows to the case where $f(z) = 0$ is an equation of the nth degree :—

If z_i is any of the points $z_1, \ldots, z_n$, and if the rest of these points are paired in any way with the points $z'_1, \ldots, z'_{n-1}$, then the sum of the angles subtended by these pairs of points at the point z_i is equal to zero (the angles being measured in each case from the unaccented to the accented member of the pair).

To which we may add :

The product of the distances from z_i to the other points $z_1, \ldots, z_n$ is n times the product of the distances from z_i to the points $z'_1, \ldots, z'_{n-1}$.

Could not the first of these propositions be brought into connection with the focal properties of the higher plane curves (see Salmon's Higher Plane Curves § 142) ?

Finally we may remark that the function $f'(z)$ being merely the first polar of the function $f(z)$ with regard to the point $z = \infty$ we can pass from the case above considered to the general case where we have to deal with the roots of the first polar of $f(z)$ with regard to any point P by means of a fractional linear transformation. Several of the above propositions can easily be enunciated so as to suit this more general case by replacing straight lines by circles through P, conics by bicircular quartics with double point at P, etc.

* We obtain incidentally the following geometrical proposition :—

The product of the lines joining the foci of the maximum ellipse with any of the vertices of the triangle is one-third the product of the sides of the triangle adjacent to this vertex.

NOVEMBER 18, 1892.